International Journal of Theoretical Physics, Vol. 37, No. 2, 1998

Finitary Algebraic Superspace

R. R. Zapatrin[1,2]

Received May 18, 1997

An algebraic scheme is suggested in which discretized spacetime turns out to be a quantum observable. As an example, a toy model producing spacetimes of four points with different topologies is presented. The possibility of incorporating this scheme into the framework of noncommutative differential geometry is discussed.

INTRODUCTION

Most schemes dealing with the quantization of gravity are built in such a way that the geometrical features of the spacetime such as metric or connection are subject to variation. Instead, on the set-theoretic and topological level the structure of the underlying spacetime remains unchanged.

It should be emphasized that there is no experimental evidence to suppose the set of all spacetime points to be given once and forever. Nobody can directly observe the points of spacetime, and therefore one should not be surprised when it happens that the entire topology of the spacetime seems different for different observers.

The concept of superspace was initially introduced by Misner *et al.* (1973) as a space whose points are spacetime geometries, so their idea could be characterized as a desire to build a classical model for the kinematics and the dynamics of spacetime geometry. In the present paper a quantum analog of superspace in which spacetime itself becomes an quantum analog of superspace in which spacetime itself becomes an observable is suggested. To put this idea on a technical footing, I confine myself to finitary spacetime substitutes, when a continuous spacetime manifold is substituted by a graph

[1] Department of Mathematics, SPb UEF, Griboyedova 30-32, 191023, St-Petersburg, Russia (address for correspondence); e-mail: zzz@russ-museum.spb.su.
[2] Division of Mathematics, Istituto per la Ricerca di Base, I-86075, Monteroduni (IS), Molise, Italy.

0020-7748/98/0300-79 9$15.00/0 © 1998 Plenum Publishing Corporation

(Finkelstein, 1996) or a finite topological space (Sorkin, 1991). It will be shown that any finite-dimensional Hilbert space can be considered a superspace if it is endowed with an additional operation of associative product.

A 'quantum room' for finite topological spaces is built in this paper. The sketch of the presented quantization scheme looks as follows. At the start we have a Hilbert space of states equipped with the additional structure of associative algebra. Then an observable is imposed, splitting the space into its eigensubspaces. The core of the scheme is the *spatialization procedure*: applied to a subspace, it manufactures a finite topological space, which is interpreted as a spacetime substitute.

The main technical result is that if the state space of a quantum system (with a finite number of degrees of freedom) is endowed with an additional structure of algebra, then with any observable property of the system we can associate a finite topological space using the spatialization procedure. Nontrivial results of this procedure occur when the algebra is noncommutative (see the example in Section 6). The results are organized as follows.

Section 1: Finitary spacetime substitutes. Overviews the discretization procedure (due to Sorkin, 1991) when continuous manifolds are replaced by finite topological spaces.

Section 2: Incidence algebras. Shows how to associate a noncommutative algebra with any finite topological space.

Section 3: The spatialization procedure. The inverse operation is realized: having a finite-dimensional algebra as its input, it furnishes a finite topological (or partially ordered) set. Applied to the incidence algebra of a poset, it restores the initial partial order up to an order isomorphism.

Section 4: Finitary algebraic superspace. Introduces the super-(state space), or, in other words, the 'quantum room' for finitary spacetime substitutes.

Section 5: Liaisons with noncommutative geometry. Some ideas are presented to show where the dynamical equations for the evolution of the spacetime topology could be taken from.

Section 6: An example. A toy model is built based on the algebra of 4×4 matrices. It is shown how different eigenstates of *one* observable are associated with the spaces of different topological structure.

1. FINITARY SPACETIME SUBSTITUTES

The coarse-graining procedure described in this section substitutes a continuous manifold by a directed graph. This procedure was introduced and described in detail by Sorkin (1991): here an alternative account of this scheme based on the notion of convergence is presented.

1.1. Formal Point of View

When we are speaking of spacetime as a manifold M, its mere definition assumes that we have a covering $\mathcal{F}$ of M by open subsets. The idea of coarse-graining is to replace the existing topology of the manifold M by that generated by the covering $\mathcal{F}$. As a result, the spacetime manifold acquires a cellular structure with respect to $\mathcal{F}$, so that the events belonging to one cell are thought of as operationally indistinguishable. Then, instead of considering the set M of all events, we can focus on its finite subset $X \subseteq M$ such that each cell contains at least one point of X.

To be more precise, consider the equivalence relation $\equiv$ on the points of M: $x \equiv y$ if and only if they belong to one cell, or, more strictly,

$$x \equiv y \quad \text{if and only if} \quad \forall \mathcal{O} \in \mathcal{F} \quad x \in \mathcal{O} \Leftrightarrow y \in \mathcal{O}$$

Taking the quotient $M/\equiv$, we obtain a T_0-space which is called the *finitary substitute* of M with respect to the covering $\mathcal{F}$. The topology on $M/\equiv$ is induced by the canonical projection $M \to M/\equiv$. Let us consider some examples.

Example 1. Let M be a piece of a plane: $M = (0, 1) \times (0, 1)$, and $\mathcal{F} = \{M, \mathcal{O}_x, \mathcal{O}_y\}$ be its covering with $\mathcal{O}_x = (0, 1/3) \times (0, 1)$ and $\mathcal{O}_y = (0, 1) \times (0, 1/3)$ (Fig. 1).

The appropriate finitary substitute is presented in Fig. 2.

Example 2. A circle (Balachandran *et al.*, 1996). Let $M = \exp(i\phi)$, and let the covering be $\mathcal{F} = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$ with

$$\mathcal{O}_1 = (-\pi/2, \pi), \qquad \mathcal{O}_2 = (\pi/2, 2\pi)$$

$$\mathcal{O}_3 = (\pi/2, \pi), \qquad \mathcal{O}_4 = (-\pi/2, 0)$$

The appropriate finitary substitute is shown in Fig. 3.

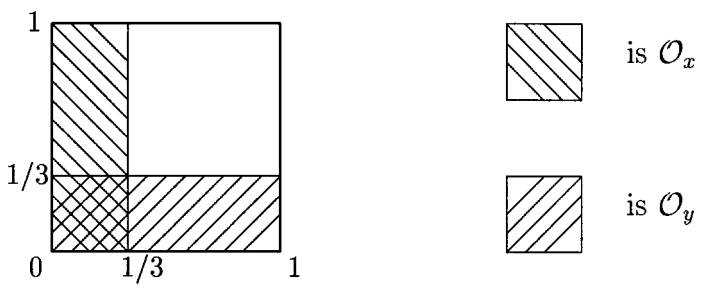

Fig. 1. The covering of a piece of plane.

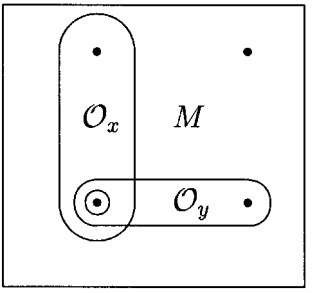

Fig. 2. A finitary substitute of the plane.

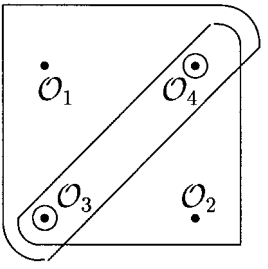

Fig. 3. A finitary substitute of the circle.

1.2. The Graphs of Finitary Substitutes[3]

Let us consider the behavior of sequences of elements in finite topological spaces. First it is worth mentioning that if a finite topological space is Hausdorf, then its topology is necessarily discrete (i.e., in a sense degenerate). Since we are going to deal with nontrivial topologies, we should not expect them to be Hausdorf. As a consequence, the theorem on the uniqueness of the limit of a sequence will no longer be valid. Consider the simplest example. Let $X = \{x, y\}$ be a set of two points with the topology depicted in Fig. 4.

Now consider the sequence $x, x, \ldots x, \ldots$ evidently having x itself as a limit point. However, by the mere definition of the limit, y is a limit point of this sequence as well! So, we also have $x, x, \ldots x, \ldots \to y$; denote it briefly $x \to y$.

Another equivalent way to define a topology τ on a set X is to define which sequences in X converge with respect to τ and which do not. So, for

Fig. 4. The set of two points with nontrivial topology and its graph of convergence.

[3] An equivalent version of the transition between graphs and finite topological spaces was presented in Sorkin (1991) and Zapatrin (1993).

Fig. 5. The graphs of convergence for the finitary substitutes of (a) a piece of plane and (b) a circle.

finite X we can instead of drawing pictures of open sets, draw the graph of convergences of sequences $x, x, \ldots x, \ldots \to y$. Figure 5 shows the convergence graphs of the above examples.

It is straightforward to prove that for any points x, y, z of a finite topological space X, $x \to y$ and $y \to z$ imply $x \to z$; therefore its graph of convergencies $G(X)$ will always be *transitive*. Note that the transitivity holds only in the case when the convergence is generated by a topology. For a more detailed account of this issue the reader is referred to Isham (1989).

1.3. From Graphs to Topological Spaces

Conversely, when we have an arbitrary graph G with the set of vertices X, we can always define a topology $\tau(G)$ on X as follows. The prebase of $\tau(G)$ is the collection of all neighborhoods of the points of X. A *neighborhood* $\mathbb{O}(x)$ of $x \in X$ is defined as the collection of all points of $y \in X$ from which x is reachable along the darts of the graph G, that is

$$\mathbb{O}(x) = \{y \in X \mid \exists y_0, \ldots, y_n \in X\colon y_i \to y_{i+1}, y_0 = y, y_n = x\}$$

If the graph G is transitive and we define the topology τ in the way described above, the convergence graph of $\tau(G)$ will be G itself. In general, the convergency graph of the topology $\tau(G)$ is the *transitive closure* of the graph G. Consider one more example. Suppose we have a graph G (Fig. 6) which is not transitive. When we consecutively pass from G to $\tau(G)$ and then to $G(\tau(G))$ we obtain its transitive closure (Fig. 6).

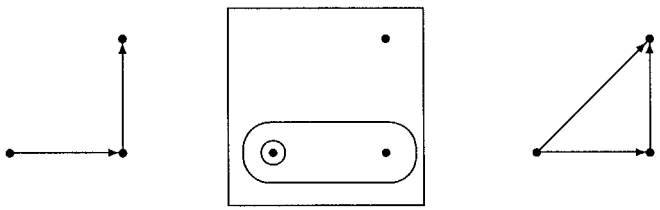

Fig. 6. The transition $G \to \tau(G) \to G(\tau(G))$.

2. INCIDENCE ALGEBRAS

2.1. Reminder on Quasiorders and Partial Orders

As was shown above, any finitary substitute can be associated with a reflexive and transitive directed graph. When such a graph is set up, we may consider its darts specifying a relation between the points of X, denote it also $\rightarrow$. This relation has the properties

$$\forall x \in X \quad x \rightarrow x$$

$$\forall x, y \in X \quad (x \rightarrow y \text{ and } y \rightarrow z) \text{ imply } x \rightarrow z \tag{1}$$

A relation on an arbitrary set having the properties (1) is called a *quasiorder*. When a quasiorder is antisymmetric

$$\forall x, y \in X \quad (x \rightarrow y \text{ and } y \rightarrow x) \text{ imply } x = z \tag{2}$$

the relation $\rightarrow$ is called a partial order. The appropriate set will be called partially ordered or, for brevity, a *poset*.

2.2. Incidence Algebras

I shall give here two equivalent definitions of the notion of incidence algebra (Rota, 1968). The first one will deal with posets in terms of graphs, and the other one directly addresses partial orders.

So, let $(X, \rightarrow)$ be a quasiordered set. Denote by G the graph of $(X, \rightarrow)$. Then consider the linear space $\mathcal{A}$ whose basis $\mathbf{e}_{ij}$ is labeled by the darts (ij) of G, i.e., by comparable pairs $i \rightarrow j$ in X. Define the product in $\mathcal{A}$ by setting it on its basic elements:

$$\mathbf{e}_{ij}\mathbf{e}_{jk} = \begin{cases} \mathbf{e}_{il} & \text{if} \quad j = k \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Note that $\mathbf{e}_{il}$ in (3) is always well defined, since G is not an arbitrary graph, but that of a partial order, which is why the existence of darts $i \rightarrow j$ and $j \rightarrow k$ always enables the existence of $i \rightarrow l$. The space $\mathcal{A}$ with the product (3) is called the *incidence algebra* of the poset $(X, \rightarrow)$.

Another equivalence definition of the incidence algebra is the following (Aigner, 1976). For a quasiordered set X define its incidence algebra $\mathcal{A}_X$, or simply $\mathcal{A}$ if no ambiguity occurs, as the collection of all complex-valued functions of two arguments vanishing on noncomparable pairs:

$$\mathcal{A} = \{a\colon X \times X \rightarrow \mathbf{C} \mid a(x, y) \neq 0 \Rightarrow x \rightarrow y\} \tag{4}$$

To make the defined linear space $\mathcal{A}$ algebra we define the product of two elements $a, b \in \mathcal{A}$ as

$$ab(x, y) = \sum_{z:\, x \to z \to y} a(x, z)b(z, y) \tag{5}$$

It can be proved that the defined product operation is associative (Rota, 1968). Since the set X is finite, the algebra $\mathcal{A}$ is a finite-dimensional associative (but not commutative, in general) algebra over $\mathbf{C}$.

Now let us clarify the meaning of the elements of $\mathcal{A}$. Let $a \in \mathcal{A}$ and x, y be two points of X. If they are not linked by a dart, then, according to (4), the value $a(x, y)$ always vanishes. So, $a(x, y)$ can be thought of as an assignment of weights (or, in other words, transition amplitudes) to the darts of the graph X. In these terms the product (5) has the following interpretation. Let $c = ab$; then $c(x, y)$ is the sum of the amplitudes of all allowed two-step transitions, the first step being ruled by a and the second by b. As an example, consider the set of two points (Fig. 4). The result of multiplication is shown on Fig. 7.

The element $c(x, y)$ of the multiple product $c = a_1 \ldots a_n$ looks similar to the Feynman sum over all paths from x to y of the length n allowed by the graph X, making them similar to S-matrices.

So, the transition from finitary substitutes to algebras is described. The inverse procedure of "spatialization" will be described in Section 3.

2.3. The Standard Matrix Representation of Incidence Algebras

Given the incidence algebra of a quasiordered set X, its standard matrix representation is obtained by choosing the basis of $\mathcal{A}$ consisting of the elements of the form $\mathbf{e}_{ab}$, with ab ranging over all ordered pairs $a \to b$ of elements of X, defined as

$$\mathbf{e}_{ab}(x, y) = \begin{cases} 1 & x = a \text{ and } y = b \text{ (provided } a \to b) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

(such matrices are called matrix units). We can also extend the range to *all* pairs of elements of X by putting $\mathbf{e}_{ab} \equiv 0$ for $a \not\to b$.[4] Then the product (5) reads

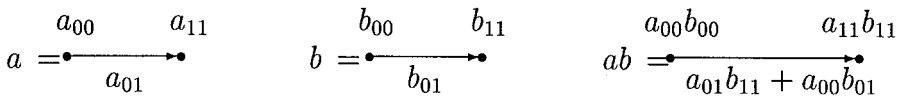

Fig. 7. Transitions on a finitary substitute.

[4] This is not strict from the algebraic point of view since the collection of such $\mathbf{e}_{ab}$ is not an ideal in the full matrix algebra. However the forthcoming results are valid for any finite topological space.

$$\mathbf{e}_{ab}\mathbf{e}_{cd} = \delta_{bc}\mathbf{e}_{ad} \tag{7}$$

With each $a \in \mathcal{A}$, the following $N \times N$ matrix (N being the cardinality of the poset X) is associated:

$$a \mapsto a_{ik} = a(x_i, x_k)$$

Let I be the incidence matrix of the graph X, that is,

$$I_{ik} = \begin{cases} 1 & x_i \to x_k \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Then the elements of $\mathcal{A}$ are represented as matrices having the following property:

$$\forall i, k \quad a_{ik} I_{ik} = a_{ik} \qquad \text{(no sum over } i, k) \tag{9}$$

The product $c = ab$ of two elements is the usual matrix product:

$$c_{ik} = c(x_i, x_k) = \sum_{i \to l \to k} a(x_i, x_l) b(x_l, x_k) = \sum_{\forall l} a_{il} b_{lk}$$

This means that we have so embedded $\mathcal{A}$ into the full matrix algebra $\mathcal{M}_N(\mathbf{C})$ that $\mathcal{A}$ is represented by the set of all matrices satisfying (9). So, to specify an incidence algebra in the standard representation we have to fix the template matrix replacing the unit entries in I_{ik}, (8), by wildcards $*$ ranging independently over all numbers. We can always reenumerate the elements of X to make I_{ik} (and hence the template matrix) an *upper-block-triangular* matrix with the blocks corresponding to cliques. In particular, when X is partially ordered, each clique contains exactly one element of X, and the incidence matrix I is upper triangular.

Examples. Let us return to our examples and build the incidence algebras associated with the finitary substitutes for the piece of plane and the circle. The template matrices for the standard matrix representation of the appropriate incidence algebras will have the form

$$I_{\text{plane}} = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}; \qquad I_{\text{circle}} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & 0 & * \end{pmatrix} \tag{10}$$

where the wildcard $*$ denotes the range over the field of numbers, for instance,

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbf{C} \right\}$$

3. THE SPATIALIZATION PROCEDURE

This section introduces the procedure which is inverse to that described in Section 2. Namely, having a finite-dimensional algebra as its input, the suggested spatialization procedure manufactures a quasiordered set. Applied to the incidence algebra $\mathcal{A}_X$ of a quasiordered space X, it yields the initial space X (up to an isomorphism).

3.1. Imploding/Exploding in Quasiorders

Let $(Y, \rightarrow)$ be a quasiordered set (1). Define the relation $\sim$ on Y

$$x \sim y \Leftrightarrow x \rightarrow y \text{ and } y \rightarrow x$$

being equivalence on Y, and consider the quotient set $X = Y/\sim$. Then X is the partially ordered set (Birkhoff, 1967). We shall call this procedure the *imploding* of a quasiorder.

When Y is treated as the graph of a finitary substitute, the transition from Y to X has the following meaning: X is obtained from Y by smashing cliques to points. Contemplating this procedure, we see that X may also be treated as the subgraph obtained from Y by deleting (except one from every clique) 'redundant' vertices with adjacent (both incoming and outgoing) darts.

We shall also consider the inverse procedure of *exploding* a partially ordered set X to a quasiorder Y. To each point of $x \in X$ a positive integer n_x is assigned. This number can be thought of as the inner dimension of infraspace (Finkelstein, 1996)—a room for gauge transformations. Then each x is replacedc by its n_x copies linked to each other by two-sided darts and linked with other vertices in the same way as x.

So, given a quasiordered set Y, we can always represent it as the partially ordered set x of its cliques equipped with the following additional structure: to each $x \in X$ an integer $n_x \geq 1$ thought of as the cardinality of the appropriate clique is assigned:

$$Y = (X, \{n_x\}) \tag{11}$$

which is illustrated in Fig. 8.

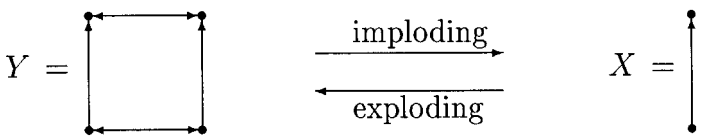

Fig. 8. Imploding of a quasiorder versus exploding a partial order.

3.2. Informal Reminder on Gel'fand Techniques

Suppose we have a commutative $*$-algebra $\mathcal{A}$ and want to represent it in the most 'natural' way, that is, by functions on a topological space M. Let us look at the algebra of continuous functions $\mathcal{A} = C(M)$. Select a point $m \in M$, and consider the collection

$$k_m = \{f \in \mathcal{A} \mid f(m) = 0\}$$

We see that k_m is an ideal in $\mathcal{A}$, and the factor algebra $\mathcal{A}/k_m$ is one dimensional, hence it is simple. The term *simple* has its strict mathematical definition: an algebra A is called simple if it has no proper (i.e., different from 0 and A) ideals. An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is called primitive if the factor algebra $\mathcal{A}/\mathcal{I}$ is simple. In the case when M is a 'good' space, there is 1-1 correspondence between the points of M and primitive ideals. So, for commutative algebras the Gel'fand spatialization procedure looks as follows:

$$\text{spat}(\mathcal{A}) = \{\text{primitive ideals}\}$$

Now, following Landi (1997), let us pass to the noncommutative case. For technical reasons I will use another, equivalent definition of primitive ideal. An ideal x of an algebra $\mathcal{A}$ is called *primitive* if it is the kernel of an irreducible representation of $\mathcal{A}$ in a vector space V_x. For finite-dimensional algebras the dimension of V_x does not depend on a particular representation of $\mathcal{A}$.

3.3. The Spatialization Procedure

A construction which builds quasiordered sets by given finite-dimensional algebras is described here. Let $\mathcal{A}$ be a subalgebra of the full matrix algebra $\text{Mat}_n(\mathbf{C})$. To build the quasiordered set associated with $\mathcal{A}$ the following is to be performed.

Step 1. Creating Cliques. Just as in the Gel'fand theory, consider the set of all primitive ideals of $\mathcal{A}$. For every such ideal x denote by n_x the dimension of the appropriate representation space V_x

$$n_x = \dim V_x$$

Then declare the set

$$X = \text{Prim}\, \mathcal{A} = \{x \mid x \text{ is a primitive ideal in } \mathcal{A}\}$$

to be the set of cliques, and the numbers n_x to be their cardinalities. So, the future finitary substitute is already created as a set (or, more precisely, as an equipped set, since we admit inner dimensions n_x of its points). It remains to endow X with the appropriate topology.

Step 2. Stretching the Elementary Darts. For every pair $x, y \in X$ we form their product xy:

$$xy = \{a \in \mathcal{A} \mid \exists u \in x,\ v \in y\colon uv = a\}$$

and their set intersection $x \cap y$. Both xy and $x \cap y$ are ideals in $\mathcal{A}$ and

$$xy \subseteq x \cap y \tag{12}$$

(since both x, y are ideals); however, the reverse inclusion may not hold. The rule I suggest is the following: the dart $x \to y$ is stretched if and only if the inclusion (12) is proper:

$$x \to y \quad \text{if and only if} \quad xy \neq x \cap y \tag{13}$$

Step 3. Forming the Partial Order. When (13) is checked for all pairs x, y, the nontransitive predecessor of the partial order on the set X is obtained. To make X partially ordered, form the transitive closure of the obtained relation:

$$\text{darts}(X) := \{(x, x)\}_{x \in X} \cup \{(x, z) \mid \exists x = y_0, \ldots, y_n = z\ Q(y_i, y_{i+1}) \neq 0\}$$

So, the finitary substitute $Y = (X, n_x)$, (11), is completely built. In the sequel denote the quasiordered set Y furnished by the spatialization procedure applied to the algebra $\mathcal{A}$ by

$$Y = \text{spat}\ \mathcal{A} \tag{14}$$

A remarkable property of the incidence algebras is the following. Applied to the incidence algebra of a quasiorder Y, this procedure restores Y up to an isomorphism of quasiorders. This was proved by Stanley (1986) for partial orders; however, his proof survives for quasiorders as well. To see how it works, consider an example.

3.4. An Example

Let us explicitly restore the quasiorder associated with the finitary substitute of the plane (Figs. 1, 2). So, the algebra $\mathcal{A}$ is the collection of matrices of the following form (10):

$$\mathcal{A} = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

where, as above, the wildcard $*$ ranges over all numbers. Now let us perform the spatialization procedure step by step.

Step 1. $\mathcal{K}$ has exactly four characters; denote their kernels by 1, 2, 3, 4:

$$1 = \ker\chi_1 = \begin{pmatrix} 0 & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}; \qquad 2 = \ker\chi_2 = \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \tag{15}$$

$$3 = \ker\chi_3 = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}; \qquad 4 = \ker\chi_4 = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the set of cliques is $X = \{1, 2, 3, 4\}$. All the cliques have dimension one since all the representations χ_i are one-dimensional.

Step 2. Now let us see how (13) works. To perform the calculations, the ordinary matrix product is used with the following arithmetics: $* \cdot * = *$; $0 \cdot * = 0$. Take two points, say 1 and 2; then

$$1 \cap 2 = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}; \qquad 1 \cdot 2 = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

so, $1 \cap 2 \neq 1 \cdot 2$, while

$$2 \cdot 1 = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} = 1 \cap 2$$

which means that we have the arrow $1 \to 2$, but not the reverse $2 \to 1$. All the rest of the calculation is performed quite analogously.

Step 3. Finally we get the following elementary darts: $1 \to 2$, $1 \to 3$, $2 \to 4$, $3 \to 4$. To complete these darts to a partial order, it remains to add only one dart $1 \to 4$ and the loops $1 \to 1, \ldots, 4 \to 4$. So, the poset (Fig. 5a) reproducing the plane is restored.

4. FINITARY ALGEBRAIC SUPERSPACE

4.1. Topokinematics

Let $\mathcal{H}$ be the state space of a physical system. If we admit that besides its usual Hilbert structure the space $\mathcal{H}$ is equipped with a product operation

$(\cdot)$: $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$, then *any* observable can be thought of as a kind of "topologimeter."[5] Namely, let **A** be an observable; denote by A the self-adjoint operator in $\mathcal{H}$ associated with it. A splits $\mathcal{H}$ into the sum of its eigensubspaces:

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \tag{16}$$

According to the projection postulate, when the measurement of **A** is performed, any state vector $\psi \in \mathcal{H}$ collapses into one of the subspaces $\mathcal{H}_i$, (16), with the probability

$$p_i = |(P_i, \psi)|^2 \tag{17}$$

where P_i is the projector onto $\mathcal{H}_i$. In the meantime, if a particular $\mathcal{H}_i$ is chosen, we may span on it a subalgebra $\mathcal{A}_i = \text{span}\mathcal{H}_i$:

$$\mathcal{A}_i = \text{span}\mathcal{H}_i = \cap\{\mathcal{H}' \mid \mathcal{H}_i \subseteq \mathcal{H}';\ \mathcal{H}' \text{ is a subalgebra of } \mathcal{H}\} \tag{18}$$

That means that measuring the observable **A**, we obtain subalgebras $\mathcal{A}_i$, (18), with probabilities p_i, (17).[6] The next step is to interpret each $\mathcal{A}_i$ as spacetime, but this is what the spatialization procedure does:

$$X_i = \text{spat}(\mathcal{A}_i) \tag{19}$$

is a finite topological space associated with the subalgebra $\mathcal{A}_i$. So, the quantum detection of topology is described by the following quantization scheme:

$$\begin{pmatrix} \text{observables} \\ + \\ \text{states} \end{pmatrix} \to \text{eigenspaces} \to \text{subalgebras} \to \text{topological spaces}$$

4.2. Two Approaches to Topodynamics

Within the suggested framework, the dynamics of topology can be introduced in two ways, similar to the Heisenberg and Schrödinger pictures in quantum mechanics.

The first approach is to fix a state $\psi \in \mathcal{H}$ and vary the constants of multiplication which make the state space $\mathcal{H}$ algebra. In this case the dynamical equations of evolution of the spacetime topology are referred to equations for these constants (note that there is a confusion in terms: in this context the constants-of-multiplication are not constants at all, being subject to variation). However, this opportunity will not be considered in this paper.

[5] The origin of the product in the state space is investigated in Parfionov and Zapatrin (1997).

[6] It should be remarked, however, that for different $\mathcal{H}_i$ their spans $\mathcal{A}_i$ may coincide or be isomorphic, so we have to adjust the correspondence $\mathcal{A}_i \to p_i$ by summing the probabilities of the isomorphic algebras.

Another approach is the following. We have to invent some topodynamical equation for the state vector ψ under the assumption that the algebraic (i.e., product) structure of the state space $\mathcal{H}$ is fixed. For the case when $\mathcal{H}$ is the full matrix algebra there is a remarkable link with the noncommutative geometry (Zapatrin, 1995) which is considered below (Section 5).

5. LIAISONS WITH NONCOMMUTATIVE GEOMETRY

In the standard general relativity the metric on a *given manifold* is subject to variation, and the equations which control it are the Einstein equations:

$$R_{ik} - \tfrac{1}{2} R g_{ik} = -G T_{ik} \tag{20}$$

It was shown by Geroch (1972) that in order to write down (20) we do not need the manifold as a set, since the entire contents of general relativity can be reformulated in terms of the algebra $\mathcal{A}$ of smooth functions on the underlying manifold. The vectors and the tensors are then introduced in mere terms of $\mathcal{A}$ treated as an algebra (Parfionov and Zapatrin, 1995). In this situation the algebra $\mathcal{A}$ is called Einstein algebra. Since it is an algebra of functions on a set it is commutative.

Then we may pass to noncommutative Einstein algebras (Heller, 1995). We can still write down equation (20), and even try to solve it for some special cases (Zapatrin, 1995), but the problem of its physical meaning immediately arises.

Now let us again return to the commutative case. Suppose equation (20) is solved and, as a result, we have the metric tensor g_{ik}. Having this tensor, we can, in turn, at each point $m \in M$ consider it as a quadratic form and split out its n eigenvectors. Having these eigenvectors, we can then restore a part of the manifold M as the exponential neighborhood of the point m. This is in a sense classical spatialization. What could be an analog of such a consecutive construction in the noncommutative case? The problem is that equation (20) entangles *vectors and tensors,* which are not the elements of the algebra $\mathcal{A}$ itself. However, when $\mathcal{A}$ is the incidence algebra of a poset, this problem can be solved.

5.1. Generalities

The idea of noncommutative geometry is to replace the (commutative) algebra of smooth functions on a manifold by a noncommutative algebra and then to build an algebraic analog of usual differential geometry (Dubois-Violette, 1981). This is possible due to the fact that the principal objects of differential geometry can be reformulated in mere terms of algebras of smooth functions. To see this, let us dwell on the vector calculus.

In standard differential geometry the vector fields on a manifold M are in 1–1 correspondence with the derivations of the algebra $\mathcal{A}$ of smooth functions on M. Recall that a derivation in an algebra $\mathcal{A}$ is a linear mapping $\mathbf{v}: \mathcal{A} \to \mathcal{A}$ enjoying the Leibniz rule:

$$\mathbf{v}(ab) = \mathbf{v}a \cdot b + a \cdot \mathbf{v}b$$

for all $a, b \in \mathcal{A}$. Denote by $\mathrm{Der}\mathcal{A}$ the set of all derivations in $\mathcal{A}$. In classical geometry the correspondence between a vector field and the appropriate derivation looks as follows:

$$\mathbf{v}f = \sum_i v^i \frac{\partial f}{\partial x^i}$$

So, the geometrical notion of a vector field can be equivalently replaced by the purely algebraic notion of derivation. The starting point to build the differential geometry could be to choose an appropriate algebra $\mathcal{A}$ called *basic algebra* (Parfionov and Zapatrin, 1995).

5.2. Noncommutative Situation

When the basic algebra $\mathcal{A}$ is noncommutative there always exists a class of derivations called *inner* ones which are associated with the elements $a \in \mathcal{A}$ in the following way: $a \mapsto \bar{a} \in \mathrm{Der}\mathcal{A}$ such that

$$\bar{a}(b) = [a, b] = ab - ba$$

The notorious property of any incidence algebra (and of full matrix algebra in particular) is that any its derivative is inner. That means that for any vector $v \in \mathrm{Der}\mathcal{A}$ there exists $\hat{v} \in \mathcal{A}$ such that for any $a \in \mathcal{A}$ the vector v acts as the commutator:

$$va = [\hat{v}, a]$$

In this case, if we obtain a solution of the Einstein equation (20) associated with a subspace V of vectors, we can immediately bring this V into the algebra $\mathcal{A}$:

$$V \mapsto \hat{V} \subseteq \mathcal{A}$$

and then directly apply the spatialization procedure described in Section 3. An example of how this can be done is given in the next section.

6. AN EXAMPLE

In this section I present an explicit example of 16-dimensional Hilbert space endowed with a product operation, and a self-adjoint operator in this

space such that for two of its different eigenspaces the spatialization procecure yields two topologically different spaces: a piece of a plane and a circle.

Let $\mathcal{H}$ be the space of all complex-valued 4×4 matrices:

$$\mathcal{H} = \mathrm{Mat}_4$$

The additional product operator on $\mathcal{H}$ will be the usual matrix product. Define the following scalar product on $\mathcal{H}$: for any $a, b \in \mathcal{H}$

$$\langle a, b \rangle = \mathrm{tr}(agb^{\dagger}) \tag{22}$$

where tr is the usual matrix trace, $(\cdot)^{\dagger}$ is the matrix transposition, and

$$g = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The metric $\langle\, , \rangle$ is nondegenerate, however indefinite: for any $a \in \mathcal{H}$ the value

$$\langle a, a \rangle = r\,\mathrm{Re}\left(\sum_i \overline{a_{i1}}(a_{i2} + a_{i3})\right) + \sum_i (|a_{i3}|^2 + |a_{i4}|^2)$$

is always real, but may have an arbitrary sign.

Consider the operator N: $\mathcal{H} \to \mathcal{H}$ defined as follows:

$$Na = an \tag{23}$$

with the following matrix n:

$$n = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The operator N is self-adjoint. To verify this, first note that $ng = gn^{\dagger}$; then

$$\langle Na, b \rangle = \mathrm{tr}(angb^{\dagger}) = \mathrm{tr}(agn^{\dagger}b^{\dagger}) = \mathrm{tr}(ag(bn)^{\dagger}) = \langle a, Nb \rangle$$

Now consider two vectors $p, c \in \mathcal{H}$:

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}; \qquad p = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Direct calculations show that the vectors p, c are eigenvectors for the operator N, (23), with the eigenvalues 1 and 2, respectively:

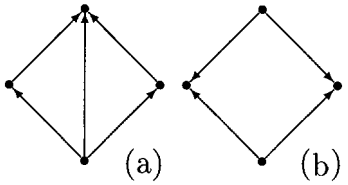

Fig. 9. (a) spat(span p); (b) spat(span c).

$$Nc = c; \qquad Np = 2p$$

Now recall that $\mathcal{H}$ has the structure of an algebra, and consider the subalgebras in $\mathcal{H}$ spanned on p and c, respectively. They are

$$\text{span } p = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}; \qquad \text{span } c = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & 0 & * \end{pmatrix}$$

having the dimensions 9 and 8, respectively. Then, if we apply the spatialization procecure (Section 3), we obtain two finite topological spaces corresponding to a piece of a plane (Fig. 9a) and to the circle (9b).

ACKNOWLEDGMENTS

The major part of the ideas exposed in this paper was born and elaborated in discussions with Prof. G. Landi, who organized my visit to the University of Trieste (November–Decembver 1995) and enabled its support from the Italian National Research Council. Partial financial support from the Russian Foundation for Fundamental Research (grant RFFI 96-02-19528) is appreciated.

REFERENCES

Aigner, M. (1976). *Higher Combinatorics,* Berlin.

Balachandran, A. P., Bimonte, G., Ercolessi, E., Landi, G., Lizzi, F., Sparano, G., and Teotonio-Sobrinho, P. (1996). Noncommutative lattices as finite approximations, *Journal of Geometry and Physics,* **18**, 163–194.

Birkhoff, G. (1967). *Lattice Theory,* Providence, Rhode Island.

Dubois-Violette, M. (1981). Calcul diffeérentiel et géometrie differentielle non-commutative, *Comptes Rendus de l'Académie de Sciences de Paris, Series I, Mathématiques,* (in press).

Finkelstein, D. (1996). *Quantum Relativity,* Springer, Berlin.

Geroch, R. (1968). What is a singularity in general relativity? *Annals of Physics,* **48**, 526.

Geroch, R. (1972). Einstein algebras, *Communications in Mathematical Physics,* **26**, 271.

Heller, M. (1995). Commutative and non-commutative Einstein algebras, *Acta Cosmologica,* **XXI-2**, 111.

Isham, C. J. (1989). An introduction to general topology and quantum topology. In *Proceedings of the NATO ASI on Physics, Geometry and Topology,* pp. 1–64.

Isham, C. J. (1994). Quantum logic and the histories approach to quantum mechanics, *Journal of Mathematical Physics,* **35**, 2157.

Landi, G. (1997). An introduction to noncommutative spaces and their geometry, eprint hep-th/9701078.

Misner, C., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation,* Freeman, San Francisco.

Parfionov, G. N., and Zapatrin, R. R. (1995). Pointless spaces in general relativity, *International Journal of Theoretical Physics,* **34**, 717.

Parfionov, G. N., and Zapatrin, R. R. (1997). Empirical topology in the histories approach to quantum theory, eprint gr-qc/9703011.

Pierce, R. P. (1982). *Associative Algebras,* Springer, Berlin.

Rota, G.-C. (1968). On the foundation of combinatorial theory, I. The theory of Möbius functions, *Zetschrift für Wahrscheinlichkeitstheorie,* **2**, 340.

Sorkin, R. D. (1991). Finitary substitutes for continuous topological spaces, *International Journal of Theoretical Physics,* **30**, 923.

Stanley, R. P. (1986). *Enumerative Combinatorics,* Wadsworth and Brooks, Monterey, California.

Zapatrin, R. R. (1993). Pre-Regge calculus: Topology via logic, *International Journal of Theoretical Physics,* **32**, 779.

Zapatrin, R. R. (1995). Matrix models for spacetime topodynamics, in *Proceedings of the ICOMM'95 (Vienna, June 3–6, 1995),* pp. 1–19; eprint gr-qc/9503066.